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A CLASS OF HYPERSURFACES WITH CONSTANT PRINCIPAL CURVATURES IN A SPHERE

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Introduction

In a series of papers [1], [2], [3], [4] E. Cartan investigated hypersurfaces M in a simply connected space form $M(c)$ of constant curvature c such that all principal curvatures of M are constant. He classified such hypersurfaces completely for the case $c \leq 0$, [1], and partially for the case $c > 0$, [2], [3], [4]. Recently H. F. Münzner [5] developed Cartan's theory and proved that to classify such hypersurfaces in a sphere is equivalent to find all homogeneous polynomials satisfying certain simultaneous differential equation. The purpose of this paper is to determine a class of M by giving a partial solution of the equation.

To state our result we shall describe an example of M in a sphere. For an integer $n \geq 2$ we denote by F_n a homogeneous polynomial

$$\left(\sum_{i=1}^{n+1} (x_i^2 - x_{i+n+1}^2)\right)^2 + 4\left(\sum_{i=1}^{n+1} x_i x_{i+n+1}\right)^2$$

of $2n + 2$ variables. Let S^{2n+1} denote the unit hypersphere in a Euclidean $(2n + 2)$-space R^{2n+2} centered at the origin. For a number t with $0 < t < \pi/4$ we denote by $M^{2n}(t)$ a hypersurface in S^{2n+1} defined by the equation

$$F_n(x) = \sin^2 2t \,, \qquad x = (x_1, \cdots, x_{2n+2}) \in S^{2n+1} \,.$$

It will be shown that $M^{2n}(t)$ is a connected compact hypersurface in S^{2n+1} having 4 constant principal curvatures with multiplicities $1, 1, n-1$ and $n-1$, and admits a transitive group of isometries. Our result can be stated as

Theorem. *Let M be a connected complete hypersurface in S^{2n+1} having 4 constant principal curvatures. If the multiplicity of one of the principal curvatures is equal to 1, then M is congruent to $M^{2n}(t)$. In particular, M admits a transitive group of isometries.*

We note that, as mentioned above, E. Cartan classified those hypersurfaces in a sphere which have at most 3 constant principal curvatures or 4 constant principal curvatures with the same multiplicity. Thus for the case $n = 2$ the above theorem is due to E. Cartan. The polynomial F_2 was first found by E. Cartan [3], and F_n by K. Nomizu [6].

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1. Differential equation

In the first place we write up all indices and their ranges used in this paper. In § 1, $\alpha, \beta = 1, \cdots, 2n+2$; $u = 1, \cdots, 2n+1$; $i, j = 1, \cdots, 2m_0 + m_1$; $r, s, t = 2m_0 + m_1 + 1, \cdots, 2n+1$, where $m_0 + m_1 = n$. In § 2, $u = 1, \cdots, 2n+1$; $i, j = 1, \cdots, 2n-1$; $r, s, t = 2n, 2n+1$; $a, b, c = 1, \cdots, n-1$. In § 3, $u = 1, \cdots, 2n+1$; $i, j = 1, \cdots, n+1$; $r, s, t = n+2, \cdots, 2n+1$.

Let M be a connected complete hypersurface in S^{2n+1} having 4 constant principal curvatures $\cot\theta_a$ $(a = 1, \cdots, 4)$ with $0 < \theta_1 < \theta_2 < \theta_3 < \theta_4 < \pi$. Let m_a be the multiplicity of $\cot\theta_a$. Then by theorems of H. F. Münzner [5, Theorems 1, 2 and 3] we know that *$m_0 = m_2$ and $m_1 = m_3$ (so $m_0 + m_1 = n \geq 2$), and that there exist a number t with $0 < t < \frac{1}{4}\pi$ and a homogeneous polynomial $\tilde{F}$ of degree 4 of $2n+2$ variables x_α such that*

$$(1.1) \qquad \sum_\alpha \left(\frac{\partial \tilde{F}}{\partial x_\alpha}\right)^2 = 16\left(\sum_\alpha x_\alpha^2\right)^3,$$

$$(1.2) \qquad \sum_\alpha \frac{\partial^2 \tilde{F}}{\partial x_\alpha^2} = 8(n - 2m_0) \sum_\alpha x_\alpha^2,$$

and $M = \{x = (x_\alpha) \in S^{2n+1}; \tilde{F}(x) = \cos 4t\}$. Conversely, for every t with $0 < t < \frac{1}{4}\pi$ and every homogeneous polynomial $\tilde{F}$ satisfying (1.1) *and* (1.2), *the set $\{x \in S^{2n+1}; \tilde{F}(x) = \cos 4t\}$ is a connected compact hypersurface in S^{2n+1} having 4 constant principal curvatures with multiplicites m_0, m_0, m_1 and m_1.*

Put $2F = (\sum_\alpha x_\alpha^2)^2 - \tilde{F}$. Then (1.1) and (1.2) are equivalent to

$$(1.3) \qquad \sum_\alpha \left(\frac{\partial F}{\partial x_\alpha}\right)^2 = 16 \sum_\alpha x_\alpha^2 F,$$

$$(1.4) \qquad \sum_\alpha \frac{\partial^2 F}{\partial x_\alpha^2} = 8(m_0 + 1) \sum_\alpha x_\alpha^2.$$

Thus in order to prove our theorem it is sufficient to prove that if $m_0 = 1$ or $m_0 = n - 1$ then every homogeneous polynomial F satisfying (1.3) and (1.4) is congruent to F_n, i.e., $F(x) = F_n(\sigma(x))$ for an orthogonal transformation σ of R^{2n+2}. In the remainder of this section we shall give the general properties of F. First fix an arbitrary index α. Without loss of generality we may assume that $F|S^{2n+1}$ takes its maximum at the point $p_\alpha = (0, \cdots, 1, \cdots, 0)$ (i.e., all the coordinates x's are zero except $x_\alpha = 1$). Then we have at p_α

$$(1.5) \qquad \frac{\partial F}{\partial x_\beta} - c x_\beta = 0 \qquad \text{for a constant } c \text{ and each } \beta.$$

Here we put $F = a_\alpha x_\alpha^4 + L x_\alpha^3 + A x_\alpha^2 + B x_\alpha + C$, where a_α, L, A, B and C denote homogeneous polynomials of $x_1, \cdots, x_{\alpha-1}, x_{\alpha+1}, \cdots, x_{2n+2}$ of degree

$0, 1, 2, 3$ and 4 respectively. From (1.5) we have $\partial L/\partial x_\beta = 0$ for $\beta \neq \alpha$ at p_α, and $c = 4a_\alpha$. From (1.3) and (1.5) it follows that $c^2 = 16a_\alpha$. These imply that $L = 0$, and $a_\alpha = 0$ or $a_\alpha = 1$. Next we shall give the relations which the polynomials A, B and C must satisfy under the assumption that $a_\alpha = 1$ for some index α, say $2n + 2$. Thus A, B and C are polynomials of $x_1, \cdots, x_{2n+1}$. From (1.3) and (1.4) we have respectively

$$\sum_u \frac{\partial^2 A}{\partial x_u^2} = 8m_0 - 4 , \tag{1.6}$$

$$\sum_u \frac{\partial^2 B}{\partial x_u^2} = 0 , \tag{1.7}$$

$$\sum_u \frac{\partial^2 C}{\partial x_u^2} + 2A = 8(m_0 + 1) \sum_u x_u^2 ; \tag{1.8}$$

$$\sum_u \left(\frac{\partial A}{\partial x_u}\right)^2 = 16 \sum_u x_u^2 , \tag{1.9}$$

$$\sum_u \frac{\partial A}{\partial x_u} \frac{\partial B}{\partial x_u} = 4B , \tag{1.10}$$

$$\sum_u \left(\frac{\partial B}{\partial x_u}\right)^2 + 2 \sum_u \frac{\partial A}{\partial x_u} \frac{\partial C}{\partial x_u} + 4A^2 = 16A \sum_u x_u^2 + 16C , \tag{1.11}$$

$$\sum_u \frac{\partial B}{\partial x_u} \frac{\partial C}{\partial x_u} + 2AB = 8B \sum_u x_u^2 , \tag{1.12}$$

$$B^2 + \sum_u \left(\frac{\partial C}{\partial x_u}\right)^2 = 16C \sum_u x_u^2 . \tag{1.13}$$

By a suitable choice of orthogonal transformation on $x_1, \cdots, x_{2n+1}$ we may set $A = \sum_u a'_u x_u^2$, $a'_1 \geq \cdots \geq a'_{2n+1}$. From (1.6) and (1.9) we have $a'^2_u = 4$ and $\sum_u a'_u = 4m_0 - 2$. Hence $a'_i = 2$ and $a'_r = -2$.

Decompose B into $P' + Q' + R' + S'$, where P', Q', R' and S' denote homogeneous polynomials of x_i and x_r whose degrees with respect to x_i are equal to $3, 2, 1$ and 0 respectively. Then taking account of the degree with respect to x_i in (1.10) and using a relation $\sum_i x_i(\partial P'/\partial x_i) = 3P'$, etc. we know $P' = R' = S' = 0$. In other words, B is of the form $4 \sum_r x_r B_r$, where B_r's denote homogeneous polynomials of x_i of degree 2.

Similarly decompose C into $P + Q + R + S + T$, where P, Q, R, S and T denote homogeneous polynomials of x_i and x_r whose degree with respect to x_i are equal to $4, 3, 2, 1$ and 0 respectively. Then we know from (1.11)

$$P = -\sum_r B_r^2 + \left(\sum_i x_i^2\right)^2 ,$$

(1.14)
$$R = \sum_i \left(\sum_r \frac{\partial B_r}{\partial x_i} x_r\right)^2 - 2 \sum_i x_i^2 \sum_r x_r^2 ,$$

$$S = 0 , \qquad T = \left(\sum_r x_r^2\right)^2 .$$

Hence (1.7), (1.8) and (1.12) are reduced respectively to

(1.15)
$$\sum_i \frac{\partial^2 B_r}{\partial x_i^2} = 0 \qquad \text{for each } r;$$

(1.16)
$$\sum_i \frac{\partial^2 Q}{\partial x_i^2} = 0 ,$$

(1.17)
$$\sum_{i,j} \left(\sum_r \frac{\partial^2 B_r}{\partial x_i \partial x_j} x_r\right)^2 = 8m_0 \sum_r x_r^2 ;$$

(1.18)
$$\sum_r B_r \frac{\partial Q}{\partial x_r} = 0 ,$$

(1.19)
$$\sum_{i,r} \frac{\partial B_r}{\partial x_i} \frac{\partial Q}{\partial x_i} x_r = 0 ,$$

(1.20)
$$\sum_{i,j,r,s,t} \frac{\partial B_r}{\partial x_i} \frac{\partial B_s}{\partial x_j} \frac{\partial^2 B_t}{\partial x_i \partial x_j} x_r x_s x_t - 8 \sum_r x_r^2 \sum_s x_s^2 = 0 .$$

From (1.13) we have

(1.21)
$$\sum_i \left(\frac{\partial P}{\partial x_i}\right)^2 + \sum_r \left(\frac{\partial Q}{\partial x_r}\right)^2 - 16P \sum_i x_i^2 = 0 .$$

Put $B_r = \sum_{i,j} b_{ij}^r x_i x_j$ and denote by B^r the symmetric matrix (b_{ij}^r) of degree $2m_0 + m_1$. Then (1.15), (1.17) and (1.20) are reduced to

(1.22) $\operatorname{trace} B^r = 0$ for each r ,

(1.23) $\operatorname{trace} (B^r)^2 = 2m_0$ for each r ,

(1.24) $\operatorname{trace} B^r B^s = 0$ for each distinct r, s ,

(1.25) $(B^r)^3 = B^r$ for each r ,

(1.26) $B^s B^r B^r + B^r B^s B^r + B^r B^r B^s = B^s$ for each distinct r, s ,

(1.27) $\mathfrak{S} B^r B^s B^t = 0$ for each mutually distinct r, s, t , where $\mathfrak{S}$ denotes the cyclic sum with respect to r, s and t. (1.27) is significant only if $m_1 \geq 2$.

Now we assert that in order to solve (1.3) and (1.4) for $m_0 = 1$ or $m_0 = n-1$ it is sufficient to consider the following two cases:

(I) $m_0 = n - 1$ and $a_\alpha = 1$ for some α ,

(II) $m_0 = 1$ and $a_\alpha = 1$ for each α .

In fact, all the possible cases besides (I) and (II) are (1) $m_0 = n-1$ and $a_\alpha = 0$ for each α, (2) $m_0 = 1$ and $a_\alpha = 0$ for each α, and (3) $m_0 = 1$ and $a_\alpha = 1$, $a_\beta = 0$ for some α, β. In any case we put $G = (\sum_\alpha x_\alpha^2)^2 - F$. Then G satisfies

$$\sum_\alpha \left(\frac{\partial G}{\partial x_\alpha}\right)^2 = 16 \sum_\alpha x_\alpha^2 G \, , \qquad \sum_\alpha \frac{\partial^2 G}{\partial x_\alpha^2} = 8(n - m_0 + 1) \sum_\alpha x_\alpha^2 \, .$$

This means that each of the cases (1), (2) and (3) is reduced to (I) or (II). We shall consider the case (I) (resp. (II)) in § 2 (resp. § 3).

2. The case (I)

We may assume that $a_{2n+2} = 1$. From (1.22), (1.23) and (1.25) it follows that by a suitable choice of orthogonal transformation on $x_1, \cdots, x_{2n-1}$ we may set $B_{2n} = \sum_a x_a^2 - \sum_a x_{a+n-1}^2$, or equivalently

$$B^{2n} = \begin{bmatrix} I & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & 0 \end{bmatrix} ,$$

where I denotes the unit matrix of degree $n-1$. Denote the transpose of a matrix J by ${}^tJ$, and put

$$B^{2n+1} = \begin{bmatrix} X & Y & u \\ {}^tY & Z & v \\ {}^tu & {}^tv & w \end{bmatrix} ,$$

where $Y = (Y_{ab})$ is a matrix of degree $n-1$, and $u = (u_a)$ and $v = (v_a)$ are column vectors. Then by (1.26) we obtain $X = Z = 0$, $w = 0$, and

$$\text{(2.1)} \qquad \sum_c Y_{ac} Y_{bc} + 2u_a u_b = \delta_{ab} \qquad \text{for each } a, b \, ,$$

$$\text{(2.2)} \qquad \sum_a u_a^2 = \sum_a v_a^2 \, .$$

Hence from (1.25) it follows that

$$u_a \sum_c Y_{bc} v_c + u_b \sum_c Y_{ac} v_c = 0 . \tag{2.3}$$

$$v_a \sum_c Y_{cb} u_c + v_b \sum_c Y_{ca} u_c = 0 \qquad \text{for each } a, b . \tag{2.4}$$

Putting $a = b$ in (2.3) we get $u_a \sum_c Y_{ac} v_c = 0$. Then by multiplying (2.3) by u_a and taking the sum over a we have $\sum_a u_a^2 \sum_c Y_{bc} v_c = 0$ for each b. Thus we need to divide our discussion into two cases.

(1) The case $\sum_a u_a^2 = 0$. It follows from (2.1) and (2.2) that $v = 0$ and Y is an orthogonal transformation on $x_n, \cdots, x_{2n-2}$. Putting $y_a = \sum_b Y_{ab} x_{b+n-1}$, we have $B_{2n+1} = 2 \sum_a x_a y_a$, $B_{2n} = \sum_a (x_a^2 - y_a^2)$ and $A = 2 \sum_a (x_a^2 + y_a^2) + x_{2n-1}^2 - \sum_r x_r^2$. Since Q is of the form $\sum_r Q_r x_r$, where Q_r's denote homogeneous polynomials of x_i of degree 3, we have, in consequence of (1.18),

$$0 = \sum_r B_r Q_r = \sum_a (x_a^2 - y_a^2) Q_{2n} + 2 \sum_a x_a y_a Q_{2n+1} .$$

Hence $Q_{2n} = B_{2n+1} L$ and $Q_{2n+1} = -B_{2n} L$ for a linear combination L of x_a, y_a and x_{2n-1}. Substituting these in (1.16) we get $\partial L / \partial x_a = \partial L / \partial y_a = 0$, i.e., $L = k x_{2n-1}$ for a constant k. Substituting P in (1.14) and the above Q in (1.21) we find $k^2 = 16$. Clearly we may adopt $k = 4$. Thus F must be of the form

$$\begin{aligned} & x_{2n+2}^4 + 2\Big(\sum_a (x_a^2 + y_a^2) + x_{2n-1}^2 - \sum_r x_r^2\Big) x_{2n+2}^2 \\ & + 4\Big(\sum_a (x_a^2 - y_a^2) x_{2n} - 2 \sum_a x_a y_a x_{2n+1}\Big) x_{2n+2} \\ & + 4 \sum_a x_a^2 \sum_a y_a^2 - 4\Big(\sum_a x_a y_a\Big)^2 + 2 \sum_a (x_a^2 + y_a^2) x_{2n-1}^2 + x_{2n-1}^4 \\ & + 4\Big(2 \sum_a x_a y_a x_{2n} + \sum_a (x_a^2 - y_a^2) x_{2n+1}\Big) x_{2n-1} \\ & + 2\Big(\sum_a (x_a^2 + y_a^2) - x_{2n-1}^2\Big) \sum_r x_r^2 + \Big(\sum_r x_r^2\Big)^2 . \end{aligned}$$

However, an orthogonal transformation $(x_1, \cdots, x_{2n+2}) \to (x_1, \cdots, x_{2n-2}, (x_{2n-1} + x_{2n})/\sqrt{2}, (x_{2n-1} - x_{2n})/\sqrt{2}, (x_{2n+1} + x_{2n+2})/\sqrt{2}, (x_{2n+1} - x_{2n+2})/\sqrt{2})$ of $\boldsymbol{R}^{2n+2}$ deforms the above polynomial into a polynomial of degree 2 with respect to each x_α. Therefore it should appear in § 3 if it is a solution.

(2) The case $\sum_a u_a^2 \neq 0$. Since $\sum_c Y_{bc} v_c = 0$ for each b, (2.2) and (2.4) imply $\sum_c Y_{ca} u_c = 0$ for each a. Multiplying (2.1) by u_b and taking the sum over b we get $2u_a \sum_b u_b^2 = u_a$ for each a. Hence $\sum_a u_a^2 = \sum_a v_a^2 = \frac{1}{2}$. It is easily seen that by a suitable choice of orthogonal transformation leaving B_{2n} invariant we may assume that $u_{n-1} = v_{n-1} = 1/\sqrt{2}$ and all the other u_a and v_a vanish. By (2.1), (2.3) and (2.4) we see that Y is of the form $\begin{bmatrix} Y' & 0 \\ 0 & 0 \end{bmatrix}$,

$Y' \in O(n-2)$. Hence

$$B_{2n} = \sum_{s=1}^{n-2} (x_s^2 - y_s^2) + x_{n-1}^2 - y_{n-1}^2 ,$$

$$B_{2n+1} = 2 \sum_{s=1}^{n-2} x_s y_s + \sqrt{2}(x_{n-1} + y_{n-1})x_{2n-1} .$$

As in the case (1), from (1.18) we have $Q_{2n} = B_{2n+1}L$ and $Q_{2n+1} = -B_{2n}L$ for a linear combination L of x_a, y_a and x_{2n-1}. Then taking account of the coefficients of x_{2n}^2 and $x_{2n}x_{2n+1}$ in (1.19) we find $Q = 0$. But substituting the first equation of (1.14) in (1.21) we can easily see $n = 2$. In fact, the coefficient of $x_{2n}x_{2n+1}$ does not vanish if $n > 2$. Since $a_\alpha = 1$ for $1 \leq \alpha \leq 6$, our polynomial should appear in § 3 if it is a solution.

3. The case (II)

We put

$$F = x_{2n+2}^4 + Ax_{2n+2}^2 + Bx_{2n+2} + C ,$$

where A, B and C denote homogeneous polynomials of $x_1, \cdots, x_{2n+1}$ of degree 2, 3 and 4 respectively. It follows from (1.22), (1.23) and (1.25) that by a suitable choice of orthogonal transformation on $x_1, \cdots, x_{n+1}$ we may set

$$B^{n+2} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} ,$$

where the central 0 denotes the zero matrix of degree $n-1$. For each $r > n+2$ we put

$$B^r = \begin{bmatrix} x^r & p^r & w^r \\ {}^t p^r & Y^r & q^r \\ w^r & {}^t q^r & z^r \end{bmatrix} ,$$

where Y^r is a symmetric matrix of degree $n-1$. Putting $r = n+2$ in (1.26) and $s = n+2$ in (1.26) we get, respectively, $x^s + z^s = 0$, $w^s = 0$, $Y^s = 0$ for each $s > n+2$, and

$$(3.1) \qquad (x^r)^2 + |p^r|^2 + |q^r|^2 = 1 , \qquad {}^t p^r \, {}^t q^r + q^r p^r = 0$$

for each $r > n+2$. From (1.25) it follows that

$$(3.2) \qquad x^r((x^r)^2 + 2|p^r|^2 - 1) = 0 , \qquad ((x^r)^2 + |p^r|^2 - 1)p^r = 0$$

for each $r > n+2$. If $n > 2$ we put $t = n+2$ in (1.27) so that

$$(3.3)\qquad {}^tp^r\,{}^tq^s + {}^tp^s\,{}^tq^r + q^rp^s + q^sp^r = 0\,,$$

$$(3.4)\qquad p^r\,{}^tp^s + {}^tq^rq^s + x^rx^s = 0 \qquad \text{for each distinct } r, s > n+2\,.$$

Lemma. *For each $r > n+2$, either $|p^r| = 1$, $q^r = 0$ and $x^r = 0$, or $p^r = 0$, $|q^r| = 1$ and $x^r = 0$.*

Proof. It follows from (3.1) and (3.2) that for each $r > n+2$, (1) $|p^r| = 1$, $q^r = 0$, $x^r = 0$, or (2) $p^r = 0$, $|q^r| = 1$, $x^r = 0$, or (3) $p^r = 0$, $q^r = 0$, $x^r = \pm 1$. Suppose that case (3) occurs, or equivalently $B^r = \pm(x_1^2 - x_{n+1}^2)$. Then such an r is unique by (1.24). Hence the polynomial P (and so also F) does not involve the term x_1^4. Since this is not the case, by the symmetry of p^r and q^r we may assume that $p^r \neq 0$ for some $r > n+2$. Then from (3.3) we have $q^sp^r = 0$ for each $s > n+2$ since $q^r = 0$ by (1). Thus $q^s = 0$ for each s. q.e.d.

Owing to this lemma and (3.4) we may set $B_r = 2x_1x_{r-n}$ for each r. Then, since $\sum_u (\partial P/\partial x_u)^2 = 16\sum_u x_u^2$, we have $\sum_r (\partial Q/\partial x_r)^2 = 0$ from (1.21). This implies that $Q = 0$. It is easily seen that the following polynomial which we just determine satisfies (1.3) and (1.4) for $m_0 = 1$:

$$x_{2n+2}^4 + 2\Big(x_1^2 + \sum_r x_r^2 - \sum_r x_{r-n}^2\Big)x_{2n+2}^2 + 8x_1\sum_r x_rx_{r-n}x_{2n+2}$$
$$+ \Big(x_1^2 + \sum_r x_{r-n}^2 - \sum_r x_r^2\Big)^2 + 4\Big(\sum_r x_rx_{r-n}\Big)^2\,.$$

This is nothing but F_n in the introduction.

4. Homogeneity of M

Let M be a hypersurface in S^{2n+1} satisfying the condition of our theorem. Then by § 1 there exist a number t with $0 < t < \frac{1}{4}\pi$ and a homogeneous polynomial F satisfying (1.3) and (1.4) such that $M = \{x \in S^{2n+1};\ F(x) = \sin^2 2t\}$, and vice versa. In § 2 we prove that every homogeneous polynomial F satisfying (1.3) and (1.4) is congruent to F_n, i.e., $F(x) = F_n(\sigma x)$ for some $\sigma \in O(2n+2)$. On the other hand, it is known [6] that a hypersurface $M^{2n}(t) = \{x \in S^{2n+1};\ F_n(x) = \sin^2 2t\}$ in S^{2n+1} admits a transitive group $G = SO(n) \times SO(2)$ of isometries, which can be considered as an analytic subgroup of $O(2n+2)$. Thus M admits a transitive group $\sigma^{-1}G\sigma$ of isometries.

Remark. There are more examples of connected compact hypersurfaces in S^{2n+1} having 4 constant principal curvatures with multiplicities m_0, m_0, m_1 and m_1 ($m_0 + m_1 = n$) (cf. [7]). We shall mention only the pairs (m_0, m_1): $(2, 2n-1)$ $(n \geq 2)$, $(4, 4n-5)$ $(n \geq 2)$, $(4,5)$ and $(6,9)$. Each of these examples admits a transitive group of isometries.

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